ON THE THEORY OF A GYROCOMPASS

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In the present paper we study the motion of a two-rotor gyrocompass. We consider the natural oscillations of the gyrocompass and its ballistic deviations which occur during the ship's maneuvers. We also study the motion of an undamped gyrocompass under the action of random forces.

1. Equations of motion of a two-rotor gyrocompass relative to a geographic reference system. A two-rotor gyrocompass [1,2] consists of two gyroscopes enclosed in a sphere called the gyrosphere which is immersed in a liquid (Fig.1) The gyroscope rotor axes are horizontal. The gyroscope housing axes are located vertically and may be rotated relative to the gyrosphere on bearings fixed on the inner surface of the gyrosphere. The housing axes of both gyroscopes are interconnected by a four-linkage mechanism — an antiparallelogram. Therefore, the rotations of both gyroscopes about their housing axes take place in opposite directions by angles which are equal in magnitude. The center link of the antiparallelogram is attached to the inner surface of the gyrosphere by two springs which try to keep the gyroscope housings in the

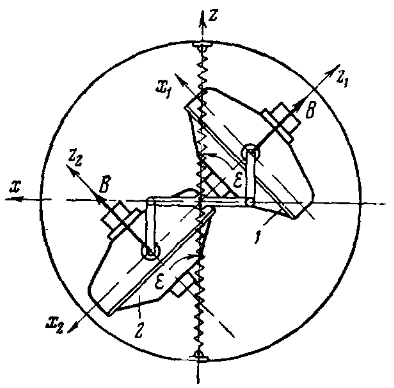

Fig. 1

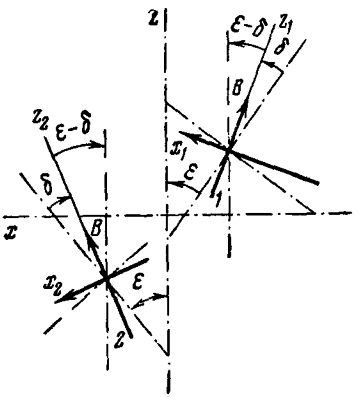

Fig. 2

positions indicated in Fig.1. The coordinate axes x, y, z (Fig.1 and 2) are rigidly fixed to the gyrosphere. The origin O coincides with the geometric center of the gyrosphere. The xz-plane in which the rotation axes of the gyroscopes are located is called the equatorial plane of the gyrosphere. The y-axis is directed upward normal to this plane. The direction of the z-axis is parallel with the bisector of the angle formed by the gyroscope rotation axes. The x-axis is perpendicular to the y- and z-axes and, together with them, forms a right-handed orthogonal trihedron xyz.

The center of gravity of the gyrosphere together with all of the elements placed in it, is located on the y-axis below the geometric center of the gyrosphere.

The location of the rotation axes of gyroscopes 1 and 2 relative to the gyrosphere can be defined by the angle δ of rotation of gyroscope 1 around its housing axis (Fig.2). With each of gyroscopes 1 and 2 is associated the coordinate axes x_i, y_i, z_i $(i = 1,2)$ with origin at the center of gravity of the corresponding gyroscope. The z_i-axis is directed along the rotation axis of the gyroscope; the y_i-axis, upward along the gyroscope housing axis; the direction of the x_i-axis is perpendicular to the y_i- and z_i-axes, and is chosen such that the trihedron $x_i y_i z_i$ is right-handed. We note that the gyroscope housing axes y_1 and y_2 and the y-axis of the gyrosphere are parallel. In Fig.1 ϵ denotes the angle which the rotation axis z_1 of gyroscope 1 makes with the z-axis of the gyrosphere at the origin when the springs are situated along the z-axis of the gyrosphere. The coordinate axes x_i, y_i, z_i $(i = 1,2)$ are the Résal axes of gyroscopes 1 and 2, respectively.

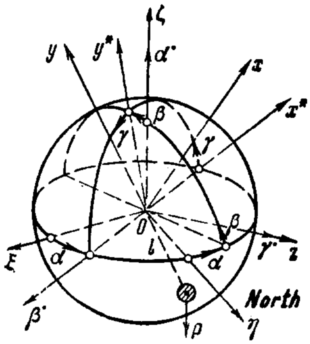

Fig. 3

As the orientation (reference) axes we choose the coordinate system $\xi\eta\zeta$ whose origin O coincides with the geometric center of the gyrosphere while the axes ξ, η, ζ, are oriented in the following manner: the ζ-axis is directed along the radius of the Earth, while the ξ- and η-axes are located in the tangent plane to the Earth's surface and directed thus: the ξ-axis to the East and the η-axis to the North.

Table 1

	ξ	η	ζ
x^*	$-\cos\alpha$	$-\sin\alpha$	0
y^*	$\sin\alpha\sin\beta$	$-\cos\alpha\sin\beta$	$\cos\beta$
z	$-\sin\alpha\cos\beta$	$\cos\alpha\cos\beta$	$\sin\beta$

The position of the gyrosphere relative to the coordinate system $\xi\eta\zeta$ is determined by the Euler angles α, β, γ (Fig.3) of which α is the angle of rotation around the ζ-axis, β is the angle of rotation around the x^*-axis lying in the horizontal plane (the angular velocity vector $\beta^{\cdot}$ is directed along the negative direction of the x^*-axis), while γ is the angle of rotation of the gyrosphere around its z-axis.

Table 1 presents the values of the cosines of the angles between the axes x^*, y^*, z (x^* is the nodal line, y^* is the transverse axis) and the axes ξ, η, ζ.

Since the dependencies

$$\mathbf{x}^0 = \mathbf{x}^{*0} \cos\gamma + \mathbf{y}^{*0} \sin\gamma, \qquad \mathbf{y}^0 = -\mathbf{x}^{*0} \sin\gamma + \mathbf{y}^{*0} \cos\gamma \qquad (1.1)$$

hold between the unit vectors of the coordinate axes x, y and x^*, y^*, the cosines of the angles between the axes x, y, z and the axes ξ, η, ζ will have the values shown in Table 2.

Table 2

	ξ	η	ζ
x	$-\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma$	$-\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma$	$\cos\beta\sin\gamma$
y	$\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma$	$\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma$	$\cos\beta\cos\gamma$
z	$-\sin\alpha\cos\beta$	$\cos\alpha\cos\beta$	$\sin\beta$

The values of the cosines of the angles between the gyrosphere axes x, z and the Résal axes x_i, z_i ($i = 1,2$) of the gyroscopes (Fig.2) are shown in Table 3.

Table 3

	x_1	z_1	x_2	z_2
x	$\cos(\varepsilon-\delta)$	$-\sin(\varepsilon-\delta)$	$\cos(\varepsilon-\delta)$	$\sin(\varepsilon-\delta)$
z	$\sin(\varepsilon-\delta)$	$\cos(\varepsilon-\delta)$	$-\sin(\varepsilon-\delta)$	$\cos(\varepsilon-\delta)$

Let us now compute the instantaneous angular velocity of the gyrosphere and the instantaneous angular velocity of each of the gyroscopes. The coordinate trihedron $\xi\eta\zeta$ (the reference system), geographically oriented as indicated above, because of the Earth's rotation and of the ship's motion on the Earth's surface, has an instantaneous angular velocity $\mathbf{u}$ whose projections on the axes ξ, η, ζ will be

$$u_1 = -\frac{v_N}{R}, \qquad u_2 = U\cos\varphi + \frac{v_E}{R}, \qquad u_3 = U\sin\varphi + \frac{v_E}{R}\tan\varphi \qquad (1.2)$$

Here U is the diurnal angular velocity of the Earth, φ is the latitude of the ship's position, v_E and v_N are the East and North components of the ship's velocity, R is the Earth's radius.

The projections of the instantaneous angular velocity of the gyrosphere on its axes x, y, z are denoted by p, q, r, respectively. They will be

$$p = \alpha^{\cdot}\cos\beta\sin\gamma - \beta^{\cdot}\cos\gamma + u_1(-\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma) +$$
$$+ u_2(-\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma) + u_3\cos\beta\sin\gamma$$
$$q = \alpha^{\cdot}\cos\beta\cos\gamma + \beta^{\cdot}\sin\gamma + u_1(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) +$$
$$+ u_2(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + u_3\cos\beta\cos\gamma$$
$$r = \alpha^{\cdot}\sin\beta + \gamma^{\cdot} - u_1\sin\alpha\cos\beta + u_2\cos\alpha\cos\beta + u_3\sin\beta \qquad (1.3)$$

The projections of the instantaneous angular velocities of each of the gyroscopes on their Résal axes x_i, y_i, z_i are denoted by p_i, q_i, r_i $(i=1,2)$, respectively. They will be

$$p_1 = p\cos(\varepsilon-\delta) + r\sin(\varepsilon-\delta), \qquad q_1 = q+\delta$$
$$r_1 = -p\sin(\varepsilon-\delta) + r\cos(\varepsilon-\delta) + \varphi_1^{\cdot} \tag{1.4}$$
$$p_2 = p\cos(\varepsilon-\delta) - r\sin(\varepsilon-\delta), \qquad q_2 = q-\delta$$
$$r_2 = p\sin(\varepsilon-\delta) + r\cos(\varepsilon-\delta) + \varphi_2$$

where $\varphi_1^{\cdot}$ and $\varphi_2^{\cdot}$ are the angular velocities of the natural rotations of gyroscopes 1 and 2, respectively. The angles α, β, γ, δ, φ_1 and φ_2 are taken as the generalized coordinates of the system under consideration.

The equations of motion of the system can be set up with the aid of the second method of Lagrange. Here we shall limit ourselves to a study of the precession motion of the system and we shall neglect its nutational oscillations. The latter is equivalent to the assumption that the kinetic moment of the whole system equals the geometric sum of the kinetic moments of the gyroscopes, while the kinetic moments of each gyroscope is directed along its natural rotation axis.

$$\mathbf{G}_i \approx C_1 r_i \mathbf{z}_i^0 \qquad (i=1,\ 2) \tag{1.5}$$

Under these assumptions the kinetic energy of the system in its motion with respect to the support point O will be determined by the approximate expression

$$T \approx {}^1/_2\, C_1 (r_1^2 + r_2^2) \tag{1.6}$$

Here C_1 denotes the moment of inertia of the gyroscope rotor with respect to the axis of its natural rotation.

The drag force moment and the active rotation moment operate around the natural rotation axes of the gyroscopes. By assuming that the total moment of the forces with respect to the axis of natural rotation of each gyroscope $Q_{\varphi_i} \equiv 0$ $(i=1,\ 2)$, and by taking into account that according to (1.6), (1.4) and (1.3)

$$\frac{\partial T}{\partial \varphi_i^{\cdot}} = C_1 r_i, \qquad \frac{\partial T}{\partial \varphi_i} = 0 \qquad (i=1,\ 2) \tag{1.7}$$

from the Lagrange equations of the second kind we obtain the relations

$$C_1 r_i = \text{const} \qquad (i=1,\ 2) \tag{1.8}$$

The natural moments of gyroscopes 1 and 2 are denoted

$$C_1 r_1 = B_1, \qquad C_1 r_2 = B_2 \tag{1.9}$$

In accordance with (1.4), the quantities r_i $(i=1,2)$ are mainly determined by the values of the angular velocities $\varphi_i^{\cdot}$ of the natural rotations of the gyroscopes since the latter are very large – of the order of 2000 sec^{-1}. For $\varphi_1^{\cdot} = \varphi_2^{\cdot}$ the difference $r_2 - r_1 = 2p\sin(\varepsilon-\delta)$ is negligibly small in comparison with r_1 and r_2. Therefore, according to (1.4), we can assume that $r_1 \approx r_2$, whence follows the equality of the natural moments of both gyroscopes

$$B_1 \approx B_2 = B \tag{1.10}$$

Let us now pass on to setting up the Lagrange equations for the noncyclic coordinates α, β, γ, δ. In accordance with (1.3), (1.4) and (1.6), the partial derivatives of the kinetic energy T with respect to the generalized velocities are

$$\frac{\partial T}{\partial \alpha^{\cdot}} = 2B \cos(\varepsilon - \delta) \sin \beta, \ \frac{\partial T}{\partial \beta^{\cdot}} = 0, \ \frac{\partial T}{\partial \gamma^{\cdot}} = 2B \cos(\varepsilon - \delta), \ \frac{\partial T}{\partial \delta^{\cdot}} = 0 \qquad (1.11)$$

The partial derivatives of the kinetic energy T with resect to the generalized coordinates have the form (1.12)

$$\partial T / \partial \alpha = 2B(-u_1 \cos\alpha \cos\beta - u_2 \sin\alpha \cos\beta) \cos(\varepsilon - \delta)$$

$$\partial T / \partial \beta = 2B(\alpha^{\cdot} \cos\beta + u_1 \sin\alpha \sin\beta -$$
$$- u_2 \cos\alpha \sin\beta + u_3 \cos\beta) \cos(\varepsilon - \delta), \qquad \partial T / \partial \gamma = 0$$

$$\partial T / \partial \delta = 2B(\alpha^{\cdot} \sin\beta + \gamma^{\cdot} - u_1 \sin\alpha \cos\beta + u_2 \cos\alpha \cos\beta + u_3 \sin\beta) \sin(\varepsilon - \delta)$$

To compute the kinetic energy (1.6) of the system, only the rotation of the coordinate trihedron $\xi\eta\zeta$ around the origin was taken into account but the motion of the origin was not considered. Therefore, to the number of external forces applied to the system there should be added the force of inertia $-m\mathbf{W}$ applied at the center of gravity of the gyrosphere, where m is the mass of the gyrosphere together with all of the elements enclosed inside it, and $\mathbf{W}$ is the acceleration of the origin of the coordinate axes $\xi\eta\zeta$, i.e. the acceleration of the point of support of the gyrocompass. The instantaneous velocity of the point of support of the gyrocompass is

$$\mathbf{v} = (RU \cos\varphi + v_E)\, \boldsymbol{\xi}^0 + v_N \boldsymbol{\eta}^0 + v_\zeta \boldsymbol{\zeta}^0 \qquad (1.13)$$

Here v_ζ denotes the vertical component of the ship's velocity. We usually take $v_\zeta = 0$. However, in certain cases, for example, in the case of ship motion on waves, it is necessary to take into account that $v_\zeta = R^{\cdot}$, where by R is meant the distance from the point of support of the gyrocompass to the Earth's center.

By taking into account that $\varphi^{\cdot} = v_N / R$, in correspondence with (1.13) the components W_1, W_2, W_3 of the acceleration of the point of support of the gyrocompass along the axes ξ, η, ζ, of the coordinate trihedron $\xi\eta\zeta$ rotating with angular velocity $\mathbf{u}$, can be represented by

$$W_1 = v_E^{\cdot} - 2v_N U \sin\varphi - \frac{v_E v_N}{R} \tan\varphi + 2v_\zeta U \cos\varphi + \frac{v_E v_\zeta}{R}$$

$$W_2 = v_N^{\cdot} + RU^2 \sin\varphi \cos\varphi + 2v_E U \sin\varphi + \frac{v_E^2}{R} \tan\varphi + \frac{v_N v_\zeta}{R}$$

$$W_3 = v_\zeta^{\cdot} - 2v_E U \cos\varphi - RU^2 \cos^2\varphi - \frac{v_E^2}{R} - \frac{v_N^2}{R} \qquad (1.14)$$

In case $v_\zeta \equiv 0$, Expressions (1.14) take the form

$$W_1 = v_E^{\cdot} - 2v_N U \sin\varphi - \frac{v_E v_N}{R} \tan\varphi$$

$$W_2 = v_N^{\cdot} + RU^2 \sin\varphi \cos\varphi + 2v_E U \sin\varphi + \frac{v_E^2}{R} \tan\varphi, \quad W_3 = -\frac{V^2}{R} \qquad (1.15)$$

where

$$V = \sqrt{(RU\cos\varphi + v_E)^2 + v_N^2} \tag{1.16}$$

The resultant of the Earth's gravitational force **P** and of the inertial force $-m\mathbf{W}$, is applied at the center of gravity of the gyrosphere and can be represented

$$\mathbf{N} = -mW_1\xi^0 - mW_2\eta^0 - (P + mW_3)\zeta^0 \tag{1.17}$$

The generalized forces along the coordinates α, β, γ, corresponding to the resultant force **N** are the partial derivatives with respect to these coordinates of the function Π_1 having the form [3]

$$\Pi_1 = -mW_1\xi_c - mW_2\eta_c - (P + mW_3)\zeta_c \tag{1.18}$$

where ξ_c, η_c, ζ_c are the coordinates of the center of gravity of the gyrosphere. By taking into account that the center of gravity of the gyrosphere lies on the y-axis at a distance l from the origin, from Table 2 we can determine the values of ξ_c, η_c, ζ_c.

Substituting these values into Expression (1.18) we bring it to the form

$$\Pi_1 = lmW_1(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma) +$$
$$+ lmW_2(\sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma) + l(P + mW_3)\cos\beta\cos\gamma \tag{1.19}$$

The potential energy of the springs which connect the center link of the antiparallelogram to the shell of the gyrosphere has the form

$$-\Pi_2 = {}^1/_2 c_1(\Delta L)^2 \tag{1.20}$$

where c_1 is the stiffness and ΔL is the deformation of the spring. From Fig.4 we see that

$$\Delta L = \sqrt{L^2 + \rho^2\sin^2\delta} - L_0 = h + \frac{\rho^2\sin^2\delta}{2L} - \frac{1}{8}\frac{\rho^4\sin^4\delta}{L^3} + \cdots \tag{1.21}$$

where L_0 is the natural length of the spring, $L - L_0 = h$ is the initial stretch of the spring.

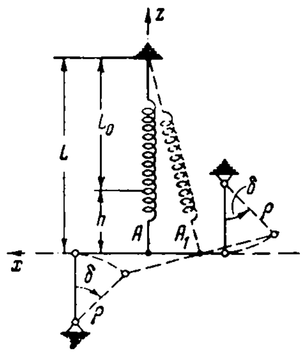

Fig. 4

Having substituted the value of ΔL into Expression (1.20) we get the following expression for the potential energy of the spring:

$$-\Pi_2 = \frac{1}{2}c_1\left(h^2 + \frac{1}{L}h\rho^2\sin^2\delta + \ldots\right) \tag{1.22}$$

where terms containing $\sin\delta$ to higher than the fourth degree are not written down and are discarded in the subsequent discussion. By denoting

$$\Pi = \Pi_1 + \Pi_2 \tag{1.23}$$

we get for the generalized forces, corresponding to the coordinates $\alpha, \beta, \gamma, \delta$, the following expressions:

$$Q_\alpha = \partial\Pi / \partial\alpha = lm\, W_1 (-\sin\alpha \sin\gamma + \cos\alpha \sin\beta \cos\gamma) + \\ + lm\, W_2 (\cos\alpha \sin\gamma + \sin\alpha \sin\beta \cos\gamma)$$

$$Q_\beta = \partial\Pi / \partial\beta = lm\, W_1 \sin\alpha \cos\beta \cos\gamma - \\ - lm\, W_2 \cos\alpha \cos\beta \cos\gamma - l(P + mW_3) \sin\beta \cos\gamma$$

$$Q_\gamma = \partial\Pi / \partial\gamma = lm\, W_1 (\cos\alpha \cos\gamma - \sin\alpha \sin\beta \sin\gamma) + \\ + lm\, W_2 (\sin\alpha \cos\gamma + \cos\alpha \sin\beta \sin\gamma) - l(P + mW_3) \cos\beta \sin\gamma$$

$$Q_\delta = \partial\Pi / \partial\delta = -\varkappa \sin\delta \cos\delta \qquad (\varkappa = L^{-1} c_1 h \rho^2) \qquad (1.24)$$

Taking (1.11), (1.12) and (1.24) into account, we obtain the equations of motion of the compass with respect to the geographically-oriented axes ξ, η, ζ

$$(1.25)$$

$$[2B \cos(\varepsilon - \delta) \sin\beta]^{\cdot} + 2B (u_1 \cos\alpha \cos\beta + \\ + u_2 \sin\alpha \cos\beta) \cos(\varepsilon - \delta) = lm\, W_1 (-\sin\alpha \sin\gamma + \\ + \cos\alpha \sin\beta \cos\gamma) + lm\, W_2 (\cos\alpha \sin\gamma + \sin\alpha \sin\beta \cos\gamma) + M_\zeta^*$$

$$2B\, [\alpha^{\cdot} \cos\beta + u_1 \sin\alpha \sin\beta - u_2 \cos\alpha \sin\beta + \\ + u_3 \cos\beta] \cos(\varepsilon - \delta) = -lm\, W_1 \sin\alpha \cos\beta \cos\gamma + \\ + lm\, W_2 \cos\alpha \cos\beta \cos\gamma + l(P + mW_3) \sin\beta \cos\gamma + M_{x^\bullet}^*$$

$$[2B \cos(\varepsilon - \delta)]^{\cdot} = lm\, W_1 (\cos\alpha \cos\gamma - \sin\alpha \sin\beta \sin\gamma) + \\ + lm\, W_2 (\sin\alpha \cos\gamma + \cos\alpha \sin\beta \sin\gamma) - l(P + mW_3) \cos\beta \sin\gamma + M_z^*$$

$$2B (\alpha^{\cdot} \sin\beta + \gamma^{\cdot} - u_1 \sin\alpha \cos\beta + u_2 \cos\alpha \cos\beta + \\ + u_3 \sin\beta) \sin(\varepsilon - \delta) = \varkappa \sin\delta \cos\delta - M_{y_1}^*$$

Here the dots over a letter or over a bracket signify the time derivative; W_i and u_i $(i = 1,2,3)$ are determined by Expressions (1.14) and (1.2); M_ζ^*, $M_{x^\bullet}^*$, M_z^*, $M_{y_1}^*$ denote the moments relative to the axes of all the remaining forces not accounted for in (1.24) which may be applied to the system.

2. Equations of motion of a gyrocompass relative to a reference system attached to the direction of absolute velocity of the ship. The motion of a gyroscopic device relative to a reference system attached to the direction of absolute velocity of the ship, was first studied in the works of Ishlinskii [4 and 5]. Let $\xi_0 \eta_0 \zeta$ be a coordinate system oriented such that one of the horizontal axes (the ξ_0-axis) is directed along the vector $\mathbf{V}$ of absolute velocity of the ship (it is assumed that the vertical component of the ship's velocity $v_\zeta \equiv 0$). As in Section 1, the ζ-axis is directed upward along the Earth's radius.

The coordinate axes ξ_0 and η_0 (Fig.5) are turned through an angle σ with respect to the axes ξ and η determined by the relations

$$\sin\sigma = v_N / V \qquad \cos\sigma = (RU \cos\varphi + v_E) / V \qquad (2.1)$$

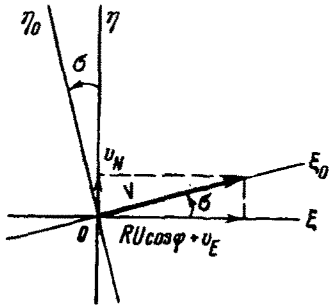

Fig. 5

The position of the gyrosphere relative to the reference system $\xi_0 \eta_0 \zeta$ is determined by the Euler angles α_1, β, γ, of which

$$\alpha_1 = \alpha - \sigma \tag{2.2}$$

while the angles β and γ are the same as those introduced in Section 1.

The projections of the instantaneous angular velocity of the trihedron $\xi_0 \eta_0 \zeta$ onto the axes ξ_0, η_0, ζ will be

$$u_1^\circ = 0, \qquad u_2^\circ = V / R, \qquad u_3^\circ = \Omega \tag{2.3}$$

$$\Omega = U \sin \varphi + (v_E / R) \tan \varphi + \sigma^{\cdot} \tag{2.4}$$

The projections of the acceleration of the point of support of the gyrocompass onto the axes ξ_0, η_0, ζ are

$$W_1^\circ = V^{\cdot}, \qquad W_2^\circ = V\Omega, \qquad W_3^\circ = -V^2 / R \tag{2.5}$$

Substituting into Equations (1.25) the quantities (2.3) and (2.5) for u_i and W_i (i = 1,2,3) we get a system of equations describing the motion of the gyrocompass relative to the axes ξ_0, η_0, ζ

$$[2B \cos(\varepsilon - \delta) \sin \beta]^{\cdot} + 2B \cos(\varepsilon - \delta)(V / R) \sin \alpha_1 \cos \beta =$$
$$= lm V^{\cdot} (-\sin \alpha_1 \sin \gamma + \cos \alpha_1 \sin \beta \cos \gamma) +$$
$$+ lm V\Omega (\cos \alpha_1 \sin \gamma + \sin \alpha_1 \sin \beta \cos \gamma) + M_\zeta^*$$

$$2B \cos(\varepsilon - \delta)[\alpha_1^{\cdot} \cos \beta - (V / R) \cos \alpha_1 \sin \beta + \Omega \cos \beta] =$$
$$= - lm V^{\cdot} \sin \alpha_1 \cos \beta \cos \gamma + lm V\Omega \cos \alpha_1 \cos \beta \cos \gamma +$$
$$+ l (P - mV^2/R) \sin \beta \cos \gamma + M_{x^*}^*$$

$$[2B \cos(\varepsilon - \delta)]^{\cdot} = lm V^{\cdot} (\cos \alpha_1 \cos \gamma - \sin \alpha_1 \sin \beta \sin \gamma) +$$
$$+ lm V\Omega (\sin \alpha_1 \cos \gamma + \cos \alpha_1 \sin \beta \sin \gamma) -$$
$$- l (P - mV^2 / R) \cos \beta \sin \gamma + M_z^*$$

$$2B \sin(\varepsilon - \delta)[\alpha_1^{\cdot} \sin \beta + \gamma^{\cdot} + (V / R) \cos \alpha_1 \cos \beta +$$
$$+ \Omega \sin \beta] = \varkappa \sin \delta \cos \delta - M_{y_1}^* \tag{2.6}$$

The equations of motion of a gyrocompass relative to axes attached to the direction of absolute velocity of the ship, obtained by Ishlinskii in [4] with the use of the theorem on kinetic moments, have the following form in our notation:

$$2B\cos(\varepsilon - \delta)[-(V/R)(\sin \alpha_1 \sin \gamma - \cos \alpha_1 \sin \beta \cos \gamma) - (\alpha_1^{\cdot} + \Omega) \cos \beta \cos \gamma - \beta^{\cdot} \sin \gamma] =$$
$$= lm V^{\cdot} \sin \alpha_1 \cos \beta - lm V\Omega \cos \alpha_1 \cos \beta - l (P - mV^2/R) \sin \beta - M_x^*$$

$$2B \cos(\varepsilon - \delta)[(V/R)(\sin \alpha_1 \cos \gamma + \cos \alpha_1 \sin \beta \sin \gamma) - (\alpha_1^{\cdot} + \Omega) \cos \beta \sin \gamma + \beta^{\cdot} \cos \gamma] = M_y^*$$

$$[2B\cos(\varepsilon - \delta)]^{\cdot} = lmV^{\cdot}(\cos \alpha_1 \cos \gamma - \sin \alpha_1 \sin \beta \sin \gamma) + lmV\Omega (\sin \alpha_1 \cos \gamma +$$
$$+ \cos \alpha_1 \sin \beta \sin \gamma) - l (P - mV^2/R) \cos \beta \sin \gamma + M_z^*$$

$$2B \sin(\varepsilon - \delta)[(V/R (\cos \alpha_1 \cos \beta + (\alpha_1^{\cdot} + \Omega) \sin \beta + \gamma^{\cdot}] = \varkappa \sin \delta \cos \delta - M_{y_1}^* \tag{2.7}$$

We can show that the systems of differential equations (2.6) and (2.7) can be obtained one from the other by means of corresponding transformations. Indeed, by multiplying the first equation in system (2.7) by $\cos\beta \sin\gamma$, the second by $\cos\beta\cos\gamma$, and the third by $\sin\beta$, and by adding these equations, we get first equation of system (2.6). By multiplying the first equation of system (2.7) by $-\cos\gamma$ and the second by $-\sin\gamma$, and by adding these equations, we get the second equation of system (2.6). The third and fourth equations of systems (2.6) and (2.7) are identical. Thus, the systems of differential equations (2.6) and (2.7) are equivalent.

3. Space gyrocompass. By a specific choice of parameters the gyrosphere of a two-rotor compass turns out to be stabilized in space and can be used as a sensitive element of a space gyrocompass or of a gyro-horizon compass.

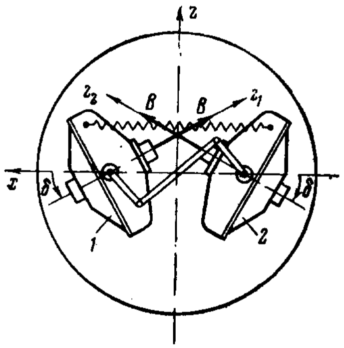

Fig. 6

The equations of motion of a space gyrocompass can be obtained from the Equations (1.25) derived above if we take it that in the device (Fig.6) the rotor axes of both gyroscopes, in the case when the rotors are not rotating, are positioned along the x-axis of the gyrosphere, to which corresponds $\epsilon = \pi/2$, while the stiffness of the spring mutually connecting the gyroscope housing is chosen as follows:

$$\varkappa = 4B^2 / lm R \tag{3.1}$$

Under the indicated conditions, for any law of motion of the ship, Equations (1.25) have the following particular solution

$$\alpha^\circ = \alpha^*, \qquad \beta^\circ = 0, \qquad \gamma^\circ = 0, \qquad \delta^\circ = \delta^* \tag{3.2}$$

where α^* and δ^* are determined from the relations

$$\sin\alpha^* = v_N / V, \qquad \cos\alpha^* = (RU\cos\varphi + v_E) / V, \qquad 2B\sin\delta^* = lm V \tag{3.3}$$

The equations of small oscillations relative to the position defined by relations (3.2) have the solution

$$w_1(t) = w_1(0)\exp\left(-i\int_0^t [\nu - \Omega(\tau)]\,d\tau\right), \qquad w_2(t) = w_2(0)\exp\left(-i\int_0^t [\nu + \Omega(\tau)]\,d\tau\right) \tag{3.4}$$

Here

$$\begin{aligned} w_1 &= \frac{V}{R}(\alpha - \alpha^*) + \nu\gamma + i\left[\nu\beta + \frac{\Xi}{lm R}(\delta - \delta^*)\right] \\ w_2 &= \frac{V}{R}(\alpha - \alpha^*) - \nu\gamma + i\left[\nu\beta - \frac{\Xi}{lm R}(\delta - \delta^*)\right] \end{aligned} \tag{3.5}$$

where

$$\nu = \sqrt{g/R}, \qquad \Xi = \sqrt{4B^2 - (lm V)^2} \tag{3.6}$$

The quantity Ω is determined from Expression (2.4) which can be reduced to the form

$$\Omega = u_3 + \alpha^{*\cdot} = V^{-2}[(RU\cos\varphi + v_E)W_2 - v_N W_1]$$

From (3.4) and (3.5) it follows that if at the initial instant $t = 0$

$$\alpha(0) = \alpha^*(0), \qquad \beta(0) = \gamma(0) = 0, \qquad \delta(0) = \delta^*(0) \tag{3.7}$$

then for any law of maneuvering of the ssip the generalized coordinates α, β, γ and δ may be varied in the following manner:

$$\alpha(t) = \alpha^* (t), \quad \beta (t) = 0, \quad \gamma (t) = 0, \quad \delta (t) = \delta^* (t) \qquad (3.8)$$

i.e. the only deviations of the device will be deviations in the azimuth $\alpha^*(t)$.

The law of motion of the space gyrocompass under arbitrary ship maneuvers, determined by Expressions (3.4), was first found in [4] by Ishlinskii in which the equations of motion of the gyrocompass were initially taken in the form (2.7).

The space gyrocompass is a high-precision device and requires great accuracy in manufacture.

The construction of a gyrocompass would require the guarantee that there were only small values of the moments of the frictional forces in the azimuthal suspension of the sensitive element in the gyrocompass, which in due course is achieved in the two-rotor gyrocompass by the use of a gyrosphere suspended in a fluid, and in other gyrocompasses by the introduction of corresponding constructive measures.

In the construction of a space gyroscope, in addition to this, we are further required to ensure sufficient smallness of the moments of the frictional forces at the supports of the gyroscope housing axes. This can be illustrated by the following example.

In the space gyrocompass constructed by the firm "Anschütz" [6] the natural moments of the gyroscope are $B = 1.5 \cdot 10^6$ gm-cm-sec , the gyroscope rotor weighs 8.8 kg , the diameter of the gyrosphere is 400 mm . At the equator when $v_E = v_N = 0$ the angle δ^* of the rotation of the gyroscope housing relative to the gyrosphere will be $\delta_0^* = 26°$. In correspondence with (3.1) and (3.3) the stiffness of the spring can be determined by the formula $\varkappa = 2B\, U/\sin\delta_0^*$, which for the device being considered amounts to about 500 gm-cm . The presence of dry frictional forces in the supports of the gyroscope housing axes gives rise to a dead zone in which the restoring moment of the spring is less than the moment of the dry frictional force. From the data mentioned above it follows that if the gyroscope dead zone is not to exceed one angular minute, it is necessary that the moment of the dry frictional force be not larger than 0.15 gm-cm . The total weight of both gyroscopes comes to about 20 kg , and thus it is a very difficult engineering problem to guarantee a very small frictional force moment at the gyroscope housing axes supports under such a weight.

The deviation α^* of the gyrocompass from the North direction, determined by relation (3.3), is called its course or velocity deviation. This deviation is computed and eliminated from the gyrocompass readings.

As shown above, for any ship maneuver law the space gyrocompass has no other deviation except velocity deviation. This is subject to the choice of the device parameters in accordance with (3.1), which guarantees the identical coincidence of the increase in the velocity deviation α^*, induced by variations of vector $\mathbf{V}$, with the additional deviations of the gyrosphere arising under the action of the inertial force $-m\mathbf{W}$. As noted, condition (3.1) is difficult to realize and as a result the stiffness $\varkappa$ of the spring in the ordinary two-rotor gyrocompass is chosen several times greater than is called for by (3.1) for a space gyrocompass. Therefore, for a maneuvering ship, besides the velocity deviations the gyrocompass has additional deviations caused by the fact that the gyrosphere displacements under the inertial force $-m\mathbf{W}$ do not coincide with the increase of velocity deviations. These additional deviations are called ballistic deviations.

4. Ballistic deviations of a gyrocompass. Variational equations. Proceeding to a study of the ballistic deviations of a gyrocompass, we introduce the following variables:

$$x_1 = \alpha - \alpha^*, \quad x_2 = \beta, \quad x_3 = \gamma, \quad x_4 = \delta - \delta^* \qquad (4.1)$$

where α^* is determined by (3.3), while δ^* satisfies the relation

$$2B\sin(\varepsilon-\delta^*)\frac{V}{R}=\varkappa\sin\delta^*\cos\delta^* \tag{4.2}$$

We note that from (2.2) and (4.1) it follows that $x_1 = a_1$ since according to (2.1) and (2.2) the relation $\sigma = \alpha^*$ holds.

The differential equations satisfied by the variables $x_1,\ldots,x_4$ can be obtained from Equations (1.25) or from Equations (2.6). Limiting ourselves to first-order terms in $x_1,\ldots,x_4$ we have the following system of differential equations:

$$\frac{V}{R}x_1+x_2-\frac{lmV}{\Xi_1}\Omega x_3=0$$

$$x_1^{\cdot}+\frac{lmV^{\cdot}}{\Xi_1}x_1-\left[\frac{l}{\Xi_1}\left(P-m\frac{V^2}{R}\right)+\frac{V}{R}\right]x_2+\frac{\Xi_2}{\Xi_1}\Omega x_4=\left(\frac{lmV}{\Xi_1}-1\right)\Omega \tag{4.3}$$

$$-\Omega x_1+\frac{1}{mV}\left(P-m\frac{V^2}{R}\right)x_3+\frac{\Xi_2}{lmV}x_4^{\cdot}+\frac{\Xi_2^{\cdot}}{lmV}x_4=\frac{1}{lmV}(\mathrm{lm}\,V^{\cdot}-\Xi_1^{\cdot})$$

$$x_3^{\cdot}+\Omega x_2-\frac{1}{\Xi_2}\left(\Xi_1\frac{V}{R}+\varkappa\cos 2\delta^*\right)x_4=0$$

Here

$$\Xi_1=2B\cos(\varepsilon-\delta^*),\qquad \Xi_2=2B\sin(\varepsilon-\delta^*) \tag{4.4}$$

The linear differential equations (4.3) with variable coefficients are variational equations.

The solution of Equations (4.3) under zero initial conditions determine the ballistic deviations of the gyrocompass. Under nonzero initial conditions the natural oscillations of the gyrocompass will also occur in addition to the ballistic deviations.

As an example, Table 4 presents the values of the functions $x_i(t)$ ($i = 1,\ldots,4$), the solutions of Equations (4.3) as obtained on an electronic computer (the author thanks A.V.Iakimenko and L.I.Gusenkova for programing the computer) under zero initial conditions for the case of a regular circulation on a course varying in accordance with the law $\psi(t)=\psi_0-\omega t$, where $\psi_0=90°$, $\omega=0.01745\ \sec^{-1}$, for a ship velocity of v = 15 m-sec^{-1}, at the three latitudes φ = 60°, 70° and 80°. The gyrocompass parameters were taken [1] as follows: $\varkappa$ = 140 gm-cm , $2B$ = 219,000 gm-cm-sec , lP = 6760 gm-cm . The angle ε was determined from the following condition:

$$[\Xi_1]_{v=0}=lmRU\cos\varphi$$

from which: ε = 45.23° when φ = 60°, ε = 62.01° when φ = 70°, ε = 76.60° when φ = 80°. From Table 4 it is seen that at the higher latitudes the deviations of the gyrocompass reach significant amounts.

Let us note that the variational equations for a space gyrocompass can be obtained from Equations (4.3). Indeed, when $\varepsilon = \pi/2$ from (3.1) and (4.2) we find that

$$\sin\delta^*=lmV/2B \tag{4.5}$$

and, consequently, for a space gyrocompass

$$\Xi_1=lmV,\qquad \Xi_2=\Xi=\sqrt{4B^2-(lmV)^2} \tag{4.6}$$

Here, as is not difficult to see, the following relations hold:

$$\frac{l}{\Xi_1}\left(P - m\frac{V^2}{R}\right) + \frac{V}{R} = \frac{P}{mV}, \qquad \frac{\Xi_2}{lmV} = -\frac{lmV^{\cdot}}{\Xi_2}$$
$$\frac{1}{\Xi_2}\left(\Xi_1\frac{V}{R} + \varkappa\cos 2\delta^*\right) = \frac{\Xi_2}{lmR} \tag{4.7}$$

In correspondence with (4.6) the right-hand sides of Equations (4.3) vanish and the variational equations for a space gyrocompass will be homogeneous differential equations.

Table 4

t sec	$10^3\,x_1$			$10^3\,x_2$		
	$\varphi = 60°$	$\varphi = 70°$	$\varphi = 80°$	$\varphi = 60°$	$\varphi = 70°$	$\varphi = 80°$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
20	1.5708	3.3003	12.1899	—0.0008	—0.0011	—0.0021
40	2.9676	6.2885	23.7681	—0.0052	—0.0059	—0.0105
60	4.0120	8.6225	33.8786	—0.0157	—0.0166	—0.0285
80	4.5392	9.9369	41.1949	—0.0274	—0.0290	—0.0520
100	4.4495	9.9233	43.8920	—0.0217	—0.0271	—0.0588
120	3.7832	8.4944	40.1603	0.0313	0.0162	—0.0061
140	2.7710	5.9707	29.6957	0.1589	0.1275	0.1546
160	1.7939	3.1104	15.4607	0.3649	0.3119	0.4378
180	1.2294	0.8333	2.9742	0.6151	0.5390	0.7878
200	1.2666	—0.2333	—3.5722	0.8444	0.7477	1.0945
220	1.8197	—0.0232	—3.7862	0.9813	0.8710	1.2515
240	2.5914	1.0607	0.1769	0.9894	0.8642	1.2020
260	3.2230	2.4330	5.7661	0.8350	0.7202	0.9473
280	3.4341	3.5743	11.0883	0.5864	0.4675	0.5341
300	3.0994	4.1532	15.0526	0.2965	0.1568	0.0348
320	2.2610	4.0467	17.1944	0.0290	—0.1558	—0.4732
340	1.0962	3.3095	17.4682	—0.1718	—0.4244	—0.9243
360	—0.1368	2.1302	16.0965	—0.2911	—0.6210	—1.2740

t sec	$10^3\,x_3$			x_4		
	$\varphi = 60°$	$\varphi = 70°$	$\varphi = 80°$	$\varphi = 60°$	$\varphi = 70°$	$\varphi = 80°$
0	0.0000	—0.0000	0.0000	0.0000	0.0000	0.0000
20	—0.0324	—0.0186	—0.0122	—0.0049	—0.0037	—0.0029
40	—0.2532	—0.1457	—0.1020	—0.0190	—0.0143	—0.0114
60	—0.8211	—0.4750	—0.3358	—0.0401	—0.0303	—0.0246
80	—1.8392	—1.0714	—0.7677	—0.0647	—0.0495	—0.0410
100	—3.3334	—1.9590	—1.4271	—0.0885	—0.0687	—0.0584
120	—5.2414	—3.1108	—2.3071	—0.1069	—0.0843	—0.0736
140	—7.4133	—4.4462	—3.3496	—0.1156	—0.0932	—0.0833
160	—9.6313	—5.8397	—4.4445	—0.1120	—0.0928	—0.0843
180	—11.6454	—7.1453	—5.4637	—0.0952	—0.0825	—0.0758
200	—13.2191	—8.2281	—6.3094	—0.0670	—0.0636	—0.0592
220	—14.1679	—8.9881	—6.9270	—0.0308	—0.0389	—0.0380
240	—14.3847	—9.3696	—7.2890	0.0086	—0.0119	—0.0154
260	—13.8498	—9.3626	—7.3858	0.0465	0.0139	0.0055
280	—12.6308	—9.0005	—7.2295	0.0784	0.0354	0.0227
300	—10.8722	—8.3555	—6.8584	0.1009	0.0505	0.0347
320	—8.7761	—7.5304	—6.3374	0.1120	0.0578	0.0405
340	—6.5752	—6.6462	—5.7523	0.1111	0.0571	0.0399
360	—4.4991	—5.8253	—5.1999	0.0994	0.0490	0.0333

5. Gyrocompass on a fixed foundation. In the case when the gyrocompass is installed on a foundation which is stationary relative to the Earth, $v_E = v_N = 0$, $\varphi = \text{const}$. Here we shall have

$$V = RU\cos\varphi, \qquad \alpha^* = 0; \qquad \Omega = U\sin\varphi \tag{5.1}$$

Relation (4.2), from which δ^* is determined, takes the form

$$2B\sin(\varepsilon - \delta^*)\, U\cos\varphi = \varkappa \sin\delta^* \cos\delta^* \tag{5.2}$$

In correspondence with (4.3) the variational equations will now be

$$\begin{gathered}
U\cos\varphi\, x_1 + x_2^{\cdot} - \frac{lm\,RU\cos\varphi}{\Xi_1}\, U\sin\varphi\, x_3 = 0 \\
x_1^{\cdot} - \left[\frac{l}{\Xi_1}(P - mRU^2\cos^2\varphi) + U\cos\varphi\right] x_2 + \frac{\Xi_2}{\Xi_1} U\sin\varphi\, x_4 = \\
= \left(\frac{lm\,RU\cos\varphi}{\Xi_1} - 1\right) U\sin\varphi \\
-U\sin\varphi\, x_1 + \frac{P - mRU^2\cos^2\varphi}{mRU\cos\varphi} x_3 + \frac{\Xi_2}{lm\,RU\cos\varphi} x_4^{\cdot} = 0 \\
x_3^{\cdot} + U\sin\varphi\, x_2 - \frac{1}{\Xi_2}(\Xi_1 U\cos\varphi + \varkappa\cos 2\delta^*)\, x_4 = 0
\end{gathered} \tag{5.3}$$

where Ξ_1 and Ξ_2 are determined by Expressions (4.4).

In order to satisfy, at least approximately, the relation (4.5), which is automatically fulfilled for a space gyrocompass at any law of maneuvering of the ship and which ensures the absence of ballistic deviations in it, the parameters for ordinary gyrocompasses are usually chosen so that on a fixed foundation

$$\Xi_1 = lm\,RU\cos\varphi \tag{5.4}$$

Condition (5.4) is ensured by different construction methods in the different types of gyrocompasses.

When condition (5.4) is fulfilled, the relation

$$\frac{l}{\Xi_1}(P - mRU^2\cos^2\varphi) + U\cos\varphi = P / mRU\cos\varphi \tag{5.5}$$

will hold.

We now introduce the new variables

$$X_1 = U\cos\varphi\, x_1, \qquad X_2 = x_2, \qquad X_3 = x_3, \qquad X_4 = (\Xi_2 / \Xi_1)\, U\cos\varphi\, x_4 \tag{5.6}$$

Denoting

$$\mu^2 = lP\Xi_2^{-2}(\varkappa\cos 2\delta^* + lm\,RU^2\cos^2\varphi) \tag{5.7}$$

and setting

$$P - mRU^2\cos^2\varphi \approx P \approx mg \tag{5.8}$$

we can reduce the variational equations (5.3) to the form

$$\begin{aligned}
&X_1 + X_2^{\cdot} - \Omega X_3 = 0, \qquad && -\Omega X_1 + \nu^2 X_3 + X_4^{\cdot} = 0 \\
&X_1^{\cdot} - \nu^2 X_2 + \Omega X_4 = 0, \qquad && \Omega X_2 + X_3^{\cdot} - \frac{\mu^2}{\nu^2} X_4 = 0
\end{aligned} \tag{5.9}$$

We note that for a space gyrocompass relation (5.2) takes the form

$$2B\cos\delta^* U\cos\varphi = (4B^2/lmR)\sin\delta^*\cos\delta^* \quad (5.10)$$

Hence it follows that

$$2B\sin\delta^* = lmR\,U\cos\varphi \quad (5.11)$$

Here, from (3.1) and (5.11) we find that

$$\varkappa\cos 2\delta^* = (lmR)^{-1}\,[4B^2 - 2\,(lmR\,U\cos\varphi)^2] \quad (5.12)$$

Having substituted into (5.7) Expression (5.12) which we have found for $\varkappa\cos 2\delta^*$, and having taken into account that in accordance with (5.11) the relation

$$\Xi_2 = 2B\cos\delta^* = \sqrt{4B^2-(lmR\,U\cos\varphi)^2}$$

holds when $\varepsilon = \pi/2$, we find that in a space gyrocompass

$$\mu^2 = \nu^2 \quad (5.13)$$

The characteristic equation of the system of differential equations (5.9) has the form

$$\lambda^4 + (\nu^2+\mu^2+2\Omega^2)\lambda^2 + \nu^2\mu^2 - (\nu^2+\mu^2)\Omega^2 + \Omega^4 = 0 \quad (5.14)$$

The roots of Equation (5.14) will be purely imaginary. Let us denote them

$$\lambda_1, \lambda_2 = \pm i\omega_1, \qquad \lambda_3, \lambda_4 = \pm i\omega_2 \quad (5.15)$$

When $\mu = \nu$, i.e. for a space gyrocompass

$$\omega_1 = \nu - \Omega, \qquad \omega_2 = \nu + \Omega \quad (5.16)$$

which is in accord with the results (3.4) derived above.

In the case when $\mu \gg \nu$, which occurs in the ordinary gyrocompass, for ω_1 and ω_2, we can get the approximate expressions

$$\omega_1 \approx \left[1 - \frac{\mu^2+3\nu^2}{2\nu^2(\mu^2-\nu^2)}\Omega^2\right]\nu \approx \nu, \qquad \omega_2 \approx \left[1 + \frac{3\mu^2+\nu^2}{2\mu^2(\mu^2-\nu^2)}\Omega^2\right]\mu \approx \mu \quad (5.17)$$

Let us go on to the construction of the solution of the system of differential equations (5.9). For greater generality let us consider the system of differential equations

$$\begin{aligned} &X_1^{\cdot} - \nu^2 X_2 + \Omega X_4 = y_1(t), \qquad && \Omega X_2 + X_3^{\cdot} - \frac{\mu^2}{\nu^2}X_4 = y_3(t) \\ &X_1 + X_2^{\cdot} - \Omega X_3 = y_2(t), && -\Omega X_1 + \nu^2 X_3 + X_4^{\cdot} = y_4(t) \end{aligned} \quad (5.18)$$

Here the functions $y_j(t)$ $(j = 1,\ldots,4)$ are defined as the external forces applied to the system. When $y_j(t) = 0$ $(j = 1,\ldots,4)$ the system of differential equations (5.18) coincides with the system of differential equations (5.9) which describe the natural oscillations of the gyrocompass.

In the original equations (1.25) and (2.6) the generalized external forces applied to the system were denoted by M_ζ^*, $M_{x^\cdot}^*$, M_z^* and $M_{y_1}^*$ respectively. As is not difficult to verify, the functions $y_j(t)$ $(j=1,\ldots,4)$ occurring in Equations (5.18) are related to the generalized external forces by the dependencies

$$y_1(t) = \frac{1}{lmR} M_{x^*}^*, \qquad y_2(t) = \frac{1}{lmRU\cos\varphi} M_{y^*}^*$$
$$y_3(t) = -\frac{1}{\Xi_2} M_{y_1}^*, \qquad y_4(t) = \frac{1}{lmR} M_z^* \tag{5.19}$$

Here $M_{y^*}^*$ is the moment of the external forces with respect to the y^*-axis, which, according to Table 1, satisfies the relation

$$M_{y^*}^* \cos\beta + M_z^* \sin\beta = M_\zeta^* \tag{5.20}$$

By introducing the matrices

$$f(D) = \begin{Vmatrix} D & -\nu^2 & 0 & \Omega \\ 1 & D & -\Omega & 0 \\ 0 & \Omega & D & -\frac{\mu^2}{\nu^2} \\ -\Omega & 0 & \nu^2 & D \end{Vmatrix}, \qquad X = \begin{Vmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Vmatrix}, \qquad y(t) = \begin{Vmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{Vmatrix} \tag{5.21}$$

where $D = d/dt$ we replace the system of scalar differential equations (5.18) by the matrix equation

$$f(D)X = y(t) \tag{5.22}$$

By (5.15) the determinant of the matrix $f(D)$ is written thus

$$\Delta(D) = (D^2 + \omega_1^2)(D^2 + \omega_2^2) \tag{5.23}$$

If $F(D)$ denotes the adjoint matrix of matrix $f(D)$, we can represent the general solution of the system of equations (5.18) in the form

$$X_j(t) = -\frac{1}{e_1}\left\{\sum_{k=1}^{4} \mathrm{Re}\,[iF_{jk}(i\omega_1)]\,X_k(0)\cos\omega_1 t - \sum_{k=1}^{4} \mathrm{Im}\,[iF_{jk}(i\omega_1)]\,X_k(0)\sin\omega_1 t\right\} + \frac{1}{e_2}\left\{\sum_{k=1}^{4} \mathrm{Re}\,[iF_{jk}(i\omega_2)]\,X_k(0)\cos\omega_2 t - \sum_{k=1}^{4} \mathrm{Im}\,[iF_{jk}(i\omega_2)]\,X_k(0)\sin\omega_2 t\right\} + \int_0^t \sum_{k=1}^{4} N_{jk}(t-\tau)\,y_k(\tau)\,d\tau \qquad (j = 1, \ldots, 4) \tag{5.24}$$

where

$$e_s = \omega_s(\omega_2^2 - \omega_1^2) \qquad (s = 1, 2) \tag{5.25}$$

and the $N_{jk}(t)$ are the elements of the weighting function matrix

$$N(t) = \|N_{jk}(t)\|$$

for the system of equations (5.18). The functions $N_{jk}(t)$ have the form

$$N_{jk}(t) = -\frac{1}{e_1}\mathrm{Re}\,[iF_{jk}(i\omega_1)\,e^{i\omega_1 t}] + \frac{1}{e_2}\mathrm{Re}\,[iF_{jk}(i\omega_2)\,e^{i\omega_2 t}] \qquad (j, k = 1, \ldots, 4) \tag{5.26}$$

We use $iF_{jk}(i\omega_s)$ $(s = 1, 2)$ to denote the elements of the matrix

$$i\,[F(D)]_{D=i\omega_s},$$

which is

$$iF(i\omega_s) = \begin{Vmatrix} A_{11}^{(s)} & iA_{12}^{(s)} & A_{13}^{(s)} & iA_{14}^{(s)} \\ iA_{21}^{(s)} & A_{22}^{(s)} & iA_{23}^{(s)} & A_{24}^{(s)} \\ A_{31}^{(s)} & iA_{32}^{(s)} & A_{33}^{(s)} & iA_{34}^{(s)} \\ iA_{41}^{(s)} & A_{42}^{(s)} & iA_{43}^{(s)} & A_{44}^{(s)} \end{Vmatrix} \tag{5.27}$$

where (5.28)

$$A_{11}^{(s)} = \omega_s^3 - (\mu^2 + \Omega^2)\,\omega_s, \qquad A_{12}^{(s)} = -\nu^2\omega_s^2 - \Omega^2\nu^2 + \mu^2\nu^2$$

$$A_{21}^{(s)} = \omega_s^2 + \Omega^2\mu^2/\nu^2 - \mu^2, \qquad A_{22}^{(s)} = A_{11}^{(s)}$$

$$A_{31}^{(s)} = -(\Omega + \Omega\mu^2/\nu^2)\,\omega_s, \qquad A_{32}^{(s)} = \Omega\omega_s^2 + \Omega\mu^2 - \Omega^3$$

$$A_{41}^{(s)} = -\Omega\omega_s^2 - \Omega\nu^2 + \Omega^3, \qquad A_{42}^{(s)} = -2\Omega\nu^2\omega_s$$

$$A_{13}^{(s)} = A_{42}^{(s)}, \qquad A_{23}^{(s)} = A_{41}^{(s)}, \qquad A_{14}^{(s)} = A_{32}^{(s)}, \qquad A_{24}^{(s)} = A_{31}^{(s)}$$

$$A_{33}^{(s)} = \omega_s^3 - (\nu^2 + \Omega^2)\,\omega_s, \qquad A_{34}^{(s)} = -\omega_s^2\mu^2/\nu^2 - \Omega^2 + \mu^2$$

$$A_{43}^{(s)} = \nu^2\omega_s^2 + \Omega^2\nu^2 - \nu^4, \qquad A_{44}^{(s)} = A_{33}^{(s)}$$

The transformation of Equations (5.18) to normal coordinates is of interest. To determine the normal coordinates it is first necessary to represent the matrix $F(i\omega_s) = [F(D)]_{D=i\omega_s}$ as the product of a column matrix and a row matrix

$$F(i\omega_s) = K(i\omega_s)\, I(i\omega_s) \qquad (s = 1, 2) \tag{5.29}$$

As can be seen from (5.27) the indicated matrices can be taken as

$$K(i\omega_s) = \begin{Vmatrix} K_1 \\ \cdots \\ K_4 \end{Vmatrix}, \qquad I(i\omega_s) = \| I_1 \ldots I_4 \| \tag{5.30}$$

where

$$K_1 = \frac{1}{2}, \qquad K_2 = i\,\frac{\omega_s^2 + \nu^2 - \Omega^2}{4\nu^2\omega_s}$$

$$K_3 = \frac{-\omega_s^2 + \nu^2 + \Omega^2}{4\Omega\nu^2}, \qquad K_4 = i\,\frac{-\omega_s^2 + \nu^2 - \Omega^2}{4\Omega\omega_s}$$

$$I_1 = 2i\,[-\omega_s^3 + (\mu^2 + \Omega^2)\,\omega_s], \qquad I_2 = 2\nu^2(-\omega_s^2 + \Omega^2 + \mu^2)$$

$$I_3 = 4i\Omega\nu^2\omega_s, \qquad I_4 = 2\Omega(\omega_s^2 + \mu^2 - \Omega^2) \tag{5.31}$$

The original coordinates $X_1, \ldots, X_4$ will be related to the normal coordinates $\xi_1, \eta_1, \xi_2, \eta_2$ by the following relations:

$$\begin{aligned} X_1 &= \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 \\ X_2 &= \frac{\omega_1^2 + \nu^2 - \Omega^2}{4\nu^2\omega_1}\eta_1 + \frac{\omega_2^2 + \nu^2 - \Omega^2}{4\nu^2\omega_2}\eta_2 \\ X_3 &= \frac{-\omega_1^2 + \nu^2 + \Omega^2}{4\Omega\nu^2}\xi_1 + \frac{-\omega_2^2 + \nu^2 + \Omega^2}{4\Omega\nu^2}\xi_2 \\ X_4 &= \frac{-\omega_1^2 + \nu^2 - \Omega^2}{4\Omega\omega_1}\eta_1 + \frac{-\omega_2^2 + \nu^2 - \Omega^2}{4\Omega\omega_2}\eta_2 \end{aligned} \tag{5.32}$$

Also, it is not difficult to get from (5.32) that

$$\xi_1 = \frac{2(\omega_2^2 - \nu^2 - \Omega^2) X_1 + 4\Omega\nu^2 X_3}{\omega_2^2 - \omega_1^2}$$

$$\eta_1 = \frac{2(-\omega_2^2 + \nu^2 - \Omega^2)\nu^2\omega_1 X_2 + 2(-\omega_2^2 - \nu^2 + \Omega^2)\Omega\omega_1 X_4}{(\omega_1^2 - \omega_2^2)(\nu^2 - \Omega^2)}$$

$$\xi_2 = \frac{2(-\omega_1^2 + \nu^2 + \Omega^2) X_1 - 4\Omega\nu^2 X_3}{\omega_2^2 - \omega_1^2}$$

$$\eta_2 = \frac{2(\omega_1^2 - \nu^2 + \Omega^2)\nu^2\omega_2 X_2 + 2(\omega_1^2 + \nu^2 - \Omega^2)\Omega\omega_2 X_4}{(\omega_1^2 - \omega_2^2)(\nu^2 - \Omega^2)} \qquad (5.33)$$

The differential equations satisfied by the normal coordinates will be

$$\frac{d\xi_1}{dt} - \omega_1\eta_1 = \frac{2(-\omega_1^2 + \mu^2 + \Omega^2)}{\omega_2^2 - \omega_1^2} y_1(t) + \frac{4\Omega\nu^2}{\omega_2^2 - \omega_1^2} y_3(t)$$

$$\frac{d\eta_1}{dt} + \omega_1\xi_1 = \frac{2\nu^2(-\omega_1^2 - \Omega^2 + \mu^2)}{\omega_1(\omega_2^2 - \omega_1^2)} y_2(t) + \frac{2\Omega(\omega_1^2 + \mu^2 - \Omega^2)}{\omega_1(\omega_2^2 - \omega_1^2)} y_4(t)$$

$$\frac{d\xi_2}{dt} - \omega_2\eta_2 = \frac{2(\omega_2^2 - \mu^2 - \Omega^2)}{\omega_2^2 - \omega_1^2} y_1(t) - \frac{4\Omega\nu^2}{\omega_2^2 - \omega_1^2} y_3(t)$$

$$\frac{d\eta_2}{dt} + \omega_2\xi_2 = \frac{2\nu^2(\omega_2^2 + \Omega^2 - \mu^2)}{\omega_2(\omega_2^2 - \omega_1^2)} y_2(t) - \frac{2\Omega(\omega_2^2 + \mu^2 - \Omega^2)}{\omega_2(\omega_2^2 - \omega_1^2)} y_4(t) \qquad (5.34)$$

Note that ehen the values (5.16), which give ω_1 and ω_2 for a space gyrocompass, are substituted into expressions (5.33) they take on the form

$$\xi_1 = X_1 + \nu X_3, \qquad \eta_1 = \nu X_2 + X_4; \qquad \xi_2 = X_1 - \nu X_3, \qquad \eta_2 = \nu X_2 - X_4 \quad (5.35)$$

Here ξ_i, η_i $(i = 1,2)$ will satisfy the differential equations which can be obtained from (5.34) if for ω_1 and ω_2 we substitute the values (5.16). These equations will be

$$\xi_1^{\cdot} - (\nu - \Omega)\eta_1 = y_1(t) + \nu y_3(t), \qquad \xi_2^{\cdot} - (\nu + \Omega)\eta_2 = y_1(t) - \nu y_3(t)$$

$$\eta_1^{\cdot} + (\nu - \Omega)\xi_1 = \nu y_2(t) + y_4(t), \qquad \eta_2^{\cdot} + (\nu + \Omega)\xi_2 = \nu y_2(t) - y_4(t) \quad (5.36)$$

6. Action of random forces on an undamped gyrocompass. Under zero initial conditions the law of motion of a gyrocompass under the action of random forces will be, according to (5.24),

$$X_j(t) = \sum_{k=1}^{4} \int_0^t N_{jk}(t - \tau) y_k(\tau) d\tau \qquad (j = 1, \ldots, 4) \qquad (6.1)$$

where the external forces $y_k(t)$ are defined by Expressions (5.19). Let

$$M_{x^*}^* = M_{y^*}^* = M_{y_1}^* = 0 \qquad (6.2)$$

and let M_z^* be some stationary random process with zero mean. Here, according to (5.19),

$$y_\rho(t) \equiv 0 \qquad (\rho = 1, 2, 3), \qquad y_4(t) = \frac{1}{lm R} M_z^* \qquad (6.3)$$

Expressions (6.1) now take on the form

$$X_j(t) = \int_0^t N_{j4}(t - \tau) y_4(\tau) d\tau \qquad (j = 1, \ldots, 4) \qquad (6.4)$$

According to (5.26) the elements of the weighting function matrix $N_{j4}(t)$ $(j = 1,\ldots,4)$ occurring in (6.4) can be put in the form

$$N_{14}(t) = a_{14}\sin\omega_1 t - b_{14}\sin\omega_2 t, \qquad N_{24}(t) = -a_{24}\cos\omega_1 t + b_{24}\cos\omega_2 t \tag{6.5}$$

$$N_{34}(t) = a_{34}\sin\omega_1 t - b_{34}\sin\omega_2 t, \qquad N_{44}(t) = -a_{44}\cos\omega_1 t + b_{44}\cos\omega_2 t$$

where

$$a_{j4} = \frac{1}{e_1}A_{j4}^{(1)}, \qquad b_{j4} = \frac{1}{e_2}A_{j4}^{(2)} \qquad (j = 1, \ldots, 4) \tag{6.6}$$

while $A_{j4}^{(s)}$ and e_s $(j = 1, \ldots, 4;\ s = 1, 2)$ are determined from Expressions (5.28) and (5.25).

The mathematical expectations of the random process $X_j(t)$ $(j = 1,\ldots,4)$ equal zero.

The variances of $X_j(t)$ $(j = 1,\ldots,4)$ are determined by

$$D_j(t) = \int_0^t\int_0^t N_{j4}(t-\tau)N_{j4}(t-\sigma)K_{44}(\tau-\sigma)\,d\tau\,d\sigma \qquad (j = 1, \ldots, 4) \tag{6.7}$$

where $K_{44}(\tau - \sigma)$ is the correlation function of the stationary random process $y_4(t)$.

For the Gaussian random process of the white noise type

$$K_{44}(\tau - \sigma) = G\delta(\tau - \sigma) \tag{6.8}$$

where $\delta(\tau - \sigma)$ is the Dirac delta-function. By (6.5) Expression (6.7) takes the form

$$D_j(t) = G\left\{\frac{a_{j4}^2 + b_{j4}^2}{2}\,t - \frac{a_{j4}b_{j4}}{\omega_1-\omega_2}\sin(\omega_1-\omega_2)t + (-1)^j\left[\frac{a_{j4}^2}{4\omega_1}\sin 2\omega_1 t + \frac{b_{j4}^2}{4\omega_2}\sin 2\omega_2 t - \frac{a_{j4}b_{j4}}{\omega_1+\omega_2}\sin(\omega_1+\omega_2)t\right]\right\} \qquad (j = 1, \ldots, 4) \tag{6.9}$$

Expressions (6.9) for the variances of the angles determining the errors of the gyrocompass contain terms which increase linearly with time. For sufficiently large values of t these terms can attain considerable magnitudes and will by themselves determine the amount of the variances.

We note that the time variation law of the signal variance at the output of the undamped system was studied for a second-order system by Sveshnikov [7].

The increase with time of the variance arises because of the absence of damping in the gyrocompass being considered here. The indicated circumstance is the reason for limiting the time interval during which a damping device is switched off.

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